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$\mathcal{PT}$-symmetric pseudo-perturbation recipe: an imaginary cubic oscillator with spikes

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Abstract

The pseudo-perturbative shifted-ℓ expansion technique (PSLET) is shown to be applicable in the non-Hermitian $\mathcal{PT}$-symmetric context. The construction of bound states for several $\mathcal{PT}$-symmetric potentials is presented, with special attention paid to $V(r) = \mathrm{i}r^3 - \alpha\sqrt{\mathrm{i}r}$ oscillators.

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1. Introduction

In their recent studies Dorey, Dunning and Tateo (DDT) [**?**] have considered the manifestly non-Hermitian Schrödinger equation, in $\hbar = 2m = 1$ units,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{r^2} - \alpha\sqrt{\mathrm{i}r} + \mathrm{i}r^3\right]\psi_{k,\ell}(r) = E_{k,\ell}\psi_{k,\ell}(x). \tag{1}$$

They have rigorously proved that the spectrum $E_{k,\ell}$ is real and discrete in the domain of sufficiently large angular momenta,

$$\ell > \max\left[\tfrac{1}{4}(2\alpha - 7), -\tfrac{1}{2}\right] \equiv \ell_{\mathrm{DDT}}(\alpha). \tag{2}$$

This inspired our subsequent study of this model [2] where we have shown that in the strong coupling regime with $\ell \gg 1$, the low-lying DDT bound states may be very well approximated by the harmonic oscillators. At the same time, we have noted that the quality of such an asymptotic approximation may deteriorate quite significantly with both the increase of excitation k and/or the decrease of ℓ.

Such a situation is, obviously, challenging. Firstly, our study [2] revealed that the manifest non-Hermiticity of the models of type (1) leads to the reliable leading order approximation *only after* we select our harmonic oscillator approximant as lying *very far* from the real axis (i.e., from the Hermitian regime). Such a recipe is, apparently, deeply incompatible with a smooth modification of the traditional zero-order approximants occurring in current Hermitian $1/\ell$ recipes (cf a small sample of some references in [3]). At the same time, the smallness of $1/\ell$ still supports the feeling that the similar perturbation techniques *should* prove efficient after their appropriate modification.

This observation offered a sufficiently strong motivation for our continued interest in the complex, non-Hermitian model (1) which may be understood as a characteristic representative of a very broad class of so-called pseudo-Hermitian models with real spectra, the analyses of which became very popular in the recent literature [4, 18]. Within this class, the strong-coupling version of DDT oscillators (1) with $\alpha \gg 1$ forms a particularly suitable testing ground as it combines the necessary reality of its spectrum with the smallness of the inverse quantity $1/\ell$. Moreover, the phenomenologically appealing non-Hermitian models such as (1) are rarely solvable in closed form so that the presence of a 'universal' small parameter $1/\ell \ll 1$ offers one of not too many ways towards their systematic approximate solution.

In section 2 we intend to discuss such a possibility in more detail.

2. Framework

The first stages of interest in the non-Hermitian oscillators (1) date back to an old paper by Caliceti *et al* [5]. It studied the imaginary cubic problem in the context of perturbation theory and, more than 20 years ago, it offered the first rigorous explanation why the spectrum in such a model may be real and discrete. In the literature, this result has been quoted as a mathematical curiosity [6] and only many years later, its possible relevance in physics re-emerged and has been emphasized [7]. This initiated an extensive discussion which resulted in the proposal of the so-called $\mathcal{PT}$-symmetric quantum mechanics by Bender and Boettcher [8].

The key idea of the new formalism lies in the empirical observation that the (phenomenologically desirable) existence of the real spectrum need not necessarily be attributed to the Hermiticity of the Hamiltonian. The current Hermiticity assumption $H = H^\dagger$ is replaced by the mere $\mathcal{PT}$ symmetry $H = H^\ddagger \equiv \mathcal{PT}H\mathcal{PT}$. Here, $\mathcal{P}$ denotes the parity ($\mathcal{P}x\mathcal{P} = -x$) while the anti-linear operator $\mathcal{T}$ mimics the time reflection ($\mathcal{T}\mathrm{i}\mathcal{T} = -\mathrm{i}$). It is easy to verify that example (1) exhibits such a type of symmetry [2] and may serve as an elementary illustration of the latter extension of quantum mechanics.

Bender and Boettcher's conjecture that $H = H^\ddagger$ may imply Im $E = 0$ is fragile. The extent as well as limitations of its validity are most easily analysed in the language of linear algebra using the biorthogonal bases [4, 9] and/or exactly solvable Hamiltonians [10]. Nevertheless, the relevance of many unsolvable oscillators originates from their applicability in physics [11] and field theory [12]. In such a setting, it is necessary to develop and test also some efficient approximation methods. New and intensive studies employed the ideas of the strong-coupling expansions [13] as well as the complex version of WKB [14], Hill determinants and Fourier transformation [15], functional analysis [16], variational and truncation techniques [17], and linear programming [18].

In what follows we intend to use the method based on the smallness of the inverse angular momentum parameter $1/\ell$. Various versions [19] of such an approach are available for Hermitian models where the combination of the central repulsive core $\ell(\ell+1)/r^2$ with a confining (i.e., asymptotically growing) interaction $V(r)$ forms a practical effective potential $V_p(r)$ which possesses a pronounced minimum. Near such a minimum the shape of the potential is naturally fitted by the elementary and solvable harmonic oscillator well. Corrections can be evaluated then in an unambiguous and systematic manner [20].

As long as we intend to move to the complex plane, the leading-order approximation may become non-unique. One finds several different complex and/or real minima of $V_p(r)$ even in our oversimplified examples (1) [2]. For all the similar non-Hermitian Hamiltonians, even the most sophisticated forms of the perturbation expansions in the powers of our small parameter $1/\ell$ lose their intuitive background and deserve careful new tests, therefore.

Some of the related reopened questions will be clarified by our forthcoming considerations inspired, basically, by the pseudo-perturbative shifted-ℓ expansion technique (PSLET) in its form designed for the standard Hermitian Hamiltonians and described, say, by Mustafa and Odeh in [20]. In this recipe, the shifts of ℓ were admitted as suitable optional auxiliary parameters while the use of the prefix 'pseudo-' just indicated that ℓ itself is an artificial, kinematical parameter rather than a genuine dynamical coupling.

3. $\mathcal{PT}$-symmetric PSLET recipe

As we already mentioned, one of the first $\mathcal{PT}$-symmetric models with an immediate impact on physics has been the Buslaev and Grecchi quartic anharmonic oscillator [7] described by the radial Schrödinger equation in d-dimensional space,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell_d(\ell_d+1)}{r^2} + V(r)\right]\chi_{k,\ell}(r) = E_{k,\ell}\chi_{k,\ell}(r). \tag{3}$$

In this model they shifted the coordinate axis to the complex plane, $r = t - \mathrm{i}c$ with a constant $\mathrm{Im}(r) = -c < 0$ and variable $\mathrm{Re}(r) = t \in (-\infty, \infty)$. They also required that $\chi_{k,\ell}(r) \in L_2(-\infty, \infty)$ at all partial waves $\ell_d = \ell + (d-3)/2$ and dimensions $d > 2$.

This example may find various sophisticated generalizations some of which will also be mentioned in due course in what follows. For example, a t-dependent shift $c = c(t)$ may be needed both for the exactly solvable Coulombic model of [21] and for all the more general and purely numerically tractable potentials $V(r) \sim -(\mathrm{i}r)^N$ of [8] with the positive exponents $N > 3$. Fortunately, the transition to $c = c(t)$ remains particularly elementary, being mediated by the mere change of the variable in equation (3) within this class (cf, e.g., [21] for an explicit illustration). Another remark might concern the assumption that the wavefunctions are square integrable. For the most elementary $\mathcal{PT}$-symmetric Hamiltonians this assumption seems very natural but in certain more sophisticated models its use may require a more careful analysis as presented, e.g., in [4, 16].

The practical experience with the Hermitian version of the pseudo-perturbation shifted-ℓ expansion technique of Mustafa and Odeh [20] may serve as a key inspiration for an appropriate complexified new PSLET recipe. Firstly, we note a formal equivalence between the assumed smallness of our parameter $1/\ell_d \approx 0$ and of its (arbitrarily) shifted form. Thus, we introduce a new symbol $\bar{l} = \ell_d - \beta$ and, simultaneously, move and rescale our coordinates $r \longrightarrow x$ to

$$x = \bar{l}^{1/2}(r - r_0). \tag{4}$$

Here r_0 is an arbitrary point, with its particular value to be determined later. Equation (3) thus becomes

$$\left\{-\bar{l}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\bar{l}^2 + (2\beta+1)\bar{l} + \beta(\beta+1)}{r_0^2[x/(r_0\bar{l}^{1/2}) + 1]^2} + \frac{\bar{l}^2}{Q}V(x(r))\right\}\Psi_{k,\ell}(x(r)) = E_{k,\ell}\Psi_{k,\ell}(x(r)) \tag{5}$$

where Q is a constant that scales the potential V at the large-ℓ_d limit and is set, for any specific choice of ℓ_d and k, equal to $\bar{l}^2$ at the end of the calculation. Expansions about this point, $x = 0$ (i.e. $r = r_0$), yield

$$\frac{1}{r_0^2[x/(r_0\bar{l}^{1/2})+1]^2} = \sum_{n=0}^{\infty}(-1)^n\frac{(n+1)}{r_0^{n+2}}x^n\bar{l}^{-n/2} \tag{6}$$

$$\frac{\bar{l}^2}{Q}V(x(r)) = \sum_{n=0}^{\infty}\left(\frac{\mathrm{d}^n V(r_0)}{\mathrm{d}r_0^n}\right)\frac{x^n}{n!Q}\bar{l}^{-(n-4)/2}. \tag{7}$$

It is also convenient to expand $E_{k,\ell}$ as

$$E_{k,\ell} = \sum_{n=-2}^{\infty}E_{k,\ell}^{(n)}\bar{l}^{-n}. \tag{8}$$

Of course, one may also consider the energy coefficient of half-entire power of $\bar{l}$ in (8) but all these coefficients vanish (cf, e.g., [19, 20]). Equation (5), therefore, reads

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\sum_{n=0}^{\infty}B_n x^n\bar{l}^{-(n-2)/2}+(2\beta+1)\sum_{n=0}^{\infty}T_n x^n\bar{l}^{-n/2}+\beta(\beta+1)\sum_{n=0}^{\infty}T_n x^n\bar{l}^{-(n+2)/2}\right\}\Psi_{k,\ell}(x)$$
$$= \left\{\sum_{n=-2}^{\infty}E_{k,\ell}^{(n)}\bar{l}^{-(n+1)}\right\}\Psi_{k,\ell}(x) \tag{9}$$

where

$$B_n = T_n + \left(\frac{\mathrm{d}^n V(r_0)}{\mathrm{d}r_0^n}\right)\frac{1}{n!Q} \qquad T_n = (-1)^n\frac{(n+1)}{r_0^{n+2}}. \tag{10}$$

Equation (9) is to be compared with the non-Hermitian $\mathcal{PT}$-symmetrized perturbed harmonic oscillator in the one-dimensional Schrödinger equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\omega^2}{4}(y-\mathrm{i}c)^2+\varepsilon_0+P(y-\mathrm{i}c)\right]\Phi_k(y) = \lambda_k\Phi_k(y) \tag{11}$$

where $P(y-\mathrm{i}c)$ is a complexified perturbation-like term and ε_0 is obviously a constant. Such a comparison implies

$$\varepsilon_0 = B_0\bar{l}+(2\beta+1)T_0+\beta(\beta+1)T_0/\bar{l}$$
$$\lambda_k = \varepsilon_0+(2k+1)\frac{\omega}{2}+\sum_{n=0}^{\infty}\lambda_k^{(n)}\bar{l}^{-(n+1)}$$
$$= B_0\bar{l}+\left[(2\beta+1)T_0+(2k+1)\frac{\omega}{2}\right]+\frac{1}{\bar{l}}\left[\beta(\beta+1)T_0+\lambda_k^{(0)}\right]+\sum_{n=2}^{\infty}\lambda_k^{(n-1)}\bar{l}^{-n}$$
$$= E_{k,\ell}^{(-2)}\bar{l}+E_{k,\ell}^{(-1)}+\sum_{n=1}^{\infty}E_{k,\ell}^{(n-1)}\bar{l}^{-n}. \tag{12}$$

The first two dominant terms are obvious

$$E_{k,\ell}^{(-2)} = \frac{1}{r_0^2}+\frac{V(r_0)}{Q} \tag{13}$$

$$E_{k,\ell}^{(-1)} = \frac{(2\beta+1)}{r_0^2}+(2k+1)\frac{\omega}{2} \tag{14}$$

and with appropriate rearrangements we obtain

$$E_{k,l}^{(0)} = \frac{\beta(\beta+1)}{r_0^2} + \lambda_k^{(0)} \tag{15}$$

$$E_{k,\ell}^{(n)} = \lambda_k^{(n)} \qquad n \geqslant 1. \tag{16}$$

Here r_0 is chosen to minimize $E_{k,\ell}^{(-2)}$, i.e.

$$\frac{\mathrm{d}E_{k,\ell}^{(-2)}}{\mathrm{d}r_0} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 E_{k,\ell}^{(-2)}}{\mathrm{d}r_0^2} > 0. \tag{17}$$

Equation (13) in turn gives, with $(\ell_d - \beta)^2 = Q$,

$$|\ell_d - \beta| = \sqrt{\frac{r_0^3 V'(r_0)}{2}}. \tag{18}$$

Consequently, $B_0 \bar{l}\big(=\bar{l}E_{k,\ell}^{(-2)}\big)$ adds a constant to the energy eigenvalues and $B_1 = 0$. The next leading correction to the energy series, $\bar{l}E_{k,\ell}^{(-1)}$, consists of a constant term and the exact eigenvalues of the unperturbed one-dimensional harmonic oscillator potential $\omega^2 x^2/4$ $(=B_2 x^2)$, where

$$0 < \omega = \omega^{(\pm)} = \pm\frac{2}{r_0^2}\Omega \qquad \Omega = \sqrt{3 + \frac{r_0 V''(r_0)}{V'(r_0)}}. \tag{19}$$

Evidently, equation (19) implies that r_0 is either pure real, $\omega = \omega^{(+)}$, or pure imaginary, $\omega = \omega^{(-)}$. Next, the shifting parameter β is determined by choosing $\bar{l}E_{k,\ell}^{(-1)} = 0$. That is

$$\beta = \beta^{(\pm)} = -\tfrac{1}{2}[1 \pm (2k+1)\Omega] \tag{20}$$

where $\beta = \beta^{(+)}$ for r_0 pure real and $\beta = \beta^{(-)}$ for r_0 pure imaginary. Then equation (9) reduces to

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{n=0}^{\infty} v^{(n)} \bar{l}^{-n/2}\right] \Psi_{k,\ell}(x) = \left[\sum_{n=-1}^{\infty} E_{k,\ell}^{(n)} \bar{l}^{-(n+1)}\right] \Psi_{k,\ell}(x) \tag{21}$$

with

$$v^{(0)}(x) = B_2 x^2 + (2\beta+1)T_0 \tag{22}$$

and for $n \geqslant 1$

$$v^{(n)}(x) = B_{n+2}x^{n+2} + (2\beta+1)T_n x^n + \beta(\beta+1)T_{n-2}x^{n-2}. \tag{23}$$

Equation (21) upon setting the wavefunctions

$$\Psi_{k,\ell}(x) = F_{k,\ell}(x)\exp(U_{k,\ell}(x))$$

readily transforms into the Riccati-type equation

$$F_{k,\ell}(x)\left[-(U''_{k,\ell}(x) + U'_{k,\ell}(x)U'_{k,\ell}(x)) + \sum_{n=0}^{\infty} v^{(n)}(x)\bar{l}^{-n/2} - \sum_{n=0}^{\infty} E_{k,\ell}^{(n-1)}\bar{l}^{-n}\right]$$
$$-2F'_{k,\ell}(x)U'_{k,\ell}(x) - F''_{k,\ell}(x) = 0$$

where the primes denote derivatives with respect to x. It is evident that this equation admits solutions (cf, e.g., [20]) of the form

$$U'_{k,\ell}(x) = \sum_{n=0}^{\infty} U_k^{(n)}(x)\bar{l}^{-n/2} + \sum_{n=0}^{\infty} G_k^{(n)}(x)\bar{l}^{-(n+1)/2} \qquad F_{k,\ell}(x) = x^k + \sum_{n=0}^{\infty}\sum_{p=0}^{k-1} a_{p,k}^{(n)} x^p \bar{l}^{-n/2}$$

with

$$U_k^{(n)}(x) = \sum_{m=0}^{n+1} D_{m,n,k} x^{2m-1} \qquad D_{0,n,k} = 0$$

$$G_k^{(n)}(x) = \sum_{m=0}^{n+1} C_{m,n,k} x^{2m}.$$

Obviously, equating the coefficients of the same powers of $\bar{l}$ and x (for each k), respectively, one can calculate the energy eigenvalues and eigenfunctions (following the uniqueness of the power series representation) from the knowledge of $C_{m,n,k}$, $D_{m,n,k}$, and $a_{p,k}^{(n)}$ in a hierarchical manner.

In order to test the performance of our strategy, let us first apply it to the two trivial, exactly solvable $\mathcal{PT}$-symmetric examples.

4. An elementary illustration of the recipe

4.1. $\mathcal{PT}$-symmetric Coulomb

Using the potential $V(r) = \mathrm{i}A/r$ (where A is a real coupling constant) in the above $\mathcal{PT}$-symmetric PSLET setting, one reveals the leading-order energy approximation

$$\bar{l}^2 E_{k,\ell}^{(-2)} = \frac{\bar{l}^2}{r_0^2} + \frac{\mathrm{i}A}{r_0}. \tag{24}$$

The unique minimum at $r_0 = 2\mathrm{i}\bar{l}^2/A$ occurs in the upper-half of the complex plane. In this case $\Omega = 1$, $\beta = \beta^{(-)} = k$, the leading energy term reads

$$\bar{l}^2 E_{k,\ell}^{(-2)} = \frac{\mathrm{i}A}{2r_0} = \frac{A^2}{4(k - \ell_d)^2} \tag{25}$$

and higher-order corrections vanish identically. Therefore, the total energy is

$$E_{n,\ell} = \frac{A^2}{4(n - 2\ell - 1)^2} \qquad n = 1, 2, 3, \ldots \tag{26}$$

where $n = k + \ell + 1$ is the principal quantum number. Evidently, the degeneracy associated with ordinary (Hermitian) Coulomb energies $E_n = -A^2/(2n)^2$ is now lifted upon the complexification of, say, the dielectric constant embedded in A. Moreover, the phenomenon of *flown away states* at $k = \ell_d$ emerges, of course if they exist at all (i.e. the probability of finding such states is presumably zero, the proof of which is already beyond our current methodical proposal). Therefore, for each k-state there is an ℓ_d-state to *fly away*.

Next, let us replace the central-like repulsive/attractive core through the transformation $\ell_d(\ell_d + 1) \to \alpha_o^2 - 1/4$, i.e. $\ell_d = -1/2 + q|\alpha_o|$, with $q = \pm 1$ denoting *quasi-parity*, and recast (25) as

$$E_{k,q} = \frac{A^2}{(2k + 1 - 2q|\alpha_o|)^2} \tag{27}$$

which is indeed the exact result obtained by Znojil and Lévai [21]. Equation (27) implies that *even-quasi-parity*, $q = +1$, states with $k = |\alpha_o| - 1/2$ *fly away* and disappear from the spectrum. Nevertheless, *quasi-parity oscillations* are now manifested by energy level crossings. That is, a state k with *even quasi-parity* crosses with a state k' with *odd quasi-parity* when $|\alpha_o| = (k - k')/2$. However, when $\alpha_o = 0$ the central-like core becomes attractive and the corresponding states cease to perform *quasi-parity oscillations*. For more details on the result (27) the reader may refer to Znojil and Lévai [21].

4.2. $\mathcal{PT}$-symmetric harmonic oscillator $V(r) = r^2$

For this potential the leading energy term reads

$$\bar{l}^2 E_{k,\ell}^{(-2)} = \frac{\bar{l}^2}{r_0^2} + r_0^2 \tag{28}$$

and supports four eligible minima (all satisfy our conditions in (17)) obtained through $r_0^4 = \bar{l}^2$ as $r_0 = \pm \mathrm{i}|\bar{l}|^{1/2}$ and $r_0 = \pm|\bar{l}|^{1/2}$. In this case $\Omega = 2$,

$$\beta = \beta^{(+)} = -(2k + 3/2) \tag{29}$$

for $r_0 = \pm|\bar{l}|^{1/2}$, and

$$\beta = \beta^{(-)} = (2k + 1/2) \tag{30}$$

for $r_0 = \pm \mathrm{i}|\bar{l}|^{1/2}$. Whilst the former (29) yields

$$\bar{l}^2 E_{k,\ell}^{(-2)} = 2r_0^2 = 4k + 2\ell_d + 3 \tag{31}$$

the latter (30) yields

$$\bar{l}^2 E_{k,\ell}^{(-2)} = 2r_0^2 = 4k + 1 - 2\ell_d. \tag{32}$$

In both cases $\beta = \beta^{(\pm)}$ the higher-order corrections vanish identically. Yet, one could combine (31) and (32) by the superscript $(\pm)$ and cast

$$E_{k,\ell}^{(\pm)} = 4k + 2 \pm (2\ell_d + 1). \tag{33}$$

Therefore, the $\mathcal{PT}$-symmetric oscillator is exactly solvable, by our recipe, and its spectrum, non-equidistant in general, exhibits some unusual features (cf [22] for more details). However, it should be noted that for the one-dimensional oscillator (where $\ell_d = -1$, and 0, even and odd parity, respectively) equation (33) implies (I) $E^{(+)}/2 = 2k + 1/2$, $E^{(-)}/2 = 2k + 3/2$ for $\ell_d = -1$ and (II) $E^{(+)}/2 = 2k + 3/2$, $E^{(-)}/2 = 2k + 1/2$ for $\ell_d = 0$ which can be combined together to form the exact well-known result

$$E_N = 2N + 1 \qquad N = 0, 1, 2, \ldots \tag{34}$$

with a new, redefined quantum number N.

5. Application: $\mathcal{PT}$-symmetric DDT oscillators

In our $\mathcal{PT}$-symmetric Schrödinger equation (1) with the practical effective potential

$$V_p(r) = \frac{\ell(\ell+1)}{r^2} - \alpha\sqrt{\mathrm{i}r} + \mathrm{i}r^3 \tag{35}$$

the general solutions themselves are analytic functions of r (cf, e.g., [6]). We may construct them in the complex plane which is cut, say, from the origin upwards. This means that $r = \xi \exp(\mathrm{i}\varphi)$ with the length $\xi \in (0, \infty)$ and the span of the angle $\varphi \in (-3\pi/2, \pi/2)$. Compact accounts of the related mathematics may be found in Bender and Boettcher [8].

Let us proceed with our $\mathcal{PT}$-PSLET and search for the minimum/minima of our leading energy term for the DDT oscillators (1)

$$\bar{l}^2 E_{k,\ell}^{(-2)} = \frac{\bar{l}^2}{r_0^2} - \alpha\sqrt{\mathrm{i}r_0} + \mathrm{i}r_0^3. \tag{36}$$

Evidently, condition (17) yields

$$r_0{}^5 + \mathrm{i}\left[\tfrac{1}{6}\alpha r_0{}^2\sqrt{\mathrm{i}r_0} + \tfrac{2}{3}\bar{l}^2\right] = 0. \tag{37}$$

Obviously, a closed form solution for this equation is hard to find (if it exists at all) and one has to appeal to numerical techniques to solve for r_0.

A priori, it is convenient to do some elementary analyses, in the vicinity of the extremes of α (mandated by condition (2)), and distinguish between the two different domains of α. For this purpose let us denote $(2\bar{l}^2)/3 = G^2$, rescale $r_0 = -\mathrm{i}|G^{2/5}|\rho$ and abbreviate $\alpha = \delta\sqrt{6G^2}$ with $0 \lesssim \delta \lesssim 1$. This gives the following new algebraic transparent form of our implicit definition of the minimum/minima in (37),

$$1 - Z = \delta\sqrt{\frac{Z}{6}} \qquad Z = \rho^5. \tag{38}$$

In the weak-coupling domain, vanishing $\delta \approx 0$, equation (38) becomes trivial $(1-Z=0)$. It is easy to verify that (36), with $\delta \approx 0$, has a *unique absolute minimum* at

$$Z = 1 \quad \Longrightarrow \quad \rho = 1 \quad \Longrightarrow \quad r_0 = -\mathrm{i}|G^{2/5}| \qquad \delta = 0. \tag{39}$$

In the strong-coupling regime, $\delta \approx 1$, equation (38) yields $Z = 2/3$.

At this point, one may choose to work with $\beta = 0$ $\big($i.e. $E^{(-1)}_{k,\ell} \neq 0\big)$ and obtain the leading (zeroth)-order approximation

$$\bar{l}^2 E^{(-2)}_{k,\ell} = \left(\frac{G^6}{Z^2}\right)^{1/5}\left[5Z - \frac{15}{2}\right] \tag{40}$$

and, with

$$\frac{\omega^2}{4} = B_2 = \frac{1}{r_0^4}\left(\frac{5}{2}\right)[1+Z] \qquad r_0 = -\mathrm{i}G^{2/5}Z^{1/5}$$

the first-order correction

$$\begin{aligned}\bar{l}E^{(-1)}_{k,\ell} &= \sqrt{\frac{3}{2}}G\left[\frac{1}{r_0^2} + (2k+1)\frac{\omega}{2}\right] \\ &= \left(\frac{G}{Z^2}\right)^{1/5}\left[\sqrt{\frac{15(1+Z)}{4}}(2k+1) - \sqrt{\frac{3}{2}}\right].\end{aligned} \tag{41}$$

Consequently, the energy series (8) reads, up to the first-order correction,

$$E_{k,\ell} = \frac{1}{Z^{2/5}}\left[G^{6/5}\left(5Z - \frac{15}{2}\right) + G^{1/5}\left(\sqrt{\frac{15(1+Z)}{4}}(2k+1) - \sqrt{\frac{3}{2}}\right)\right]. \tag{42}$$

Nevertheless, one may choose to work with $\beta = \beta^{(-)} \neq 0$ (i.e. $E^{(-1)}_{k,\ell} = 0$) and obtain

$$\beta = \beta^{(-)} = -\tfrac{1}{2}[1 - (2k+1)\sqrt{5(1+Z)/2}]. \tag{43}$$

Thus the zeroth-order approximation yields

$$\bar{l}^2 E^{(-2)}_{k,\ell} = \left(\frac{G_s^6}{Z^2}\right)^{1/5}\left[5Z - \frac{15}{2}\right] \qquad G_s = \sqrt{\frac{2}{3}}\left[\ell_d + \frac{1}{2} - (2k+1)\sqrt{\frac{5(1+Z)}{8}}\right]. \tag{44}$$

In figure 1 we plot the energies of (44) versus $\ell \in (-5, 5)$ at different values of $Z = Z(\delta) \in (2/3, 1)$. Obviously, our results show that even with $\ell < \ell_{\mathrm{DDT}}(\alpha)$ the spectrum remains real and discrete. Moreover, once we replace $\ell(\ell+1) \longrightarrow \alpha_o^2 - 1/4$, i.e. $\ell \longrightarrow -1/2 + q|\alpha_o|$ with $q = \pm 1$ denoting quasi-parity, *quasi-parity oscillations* are manifested by the unavoidable energy level crossings (see figure 2).

Table 1 shows that our results from equations (42) and (44) compare satisfactorily with those obtained by Znojil *et al* [2], via direct variable representation (DVR). We may mention

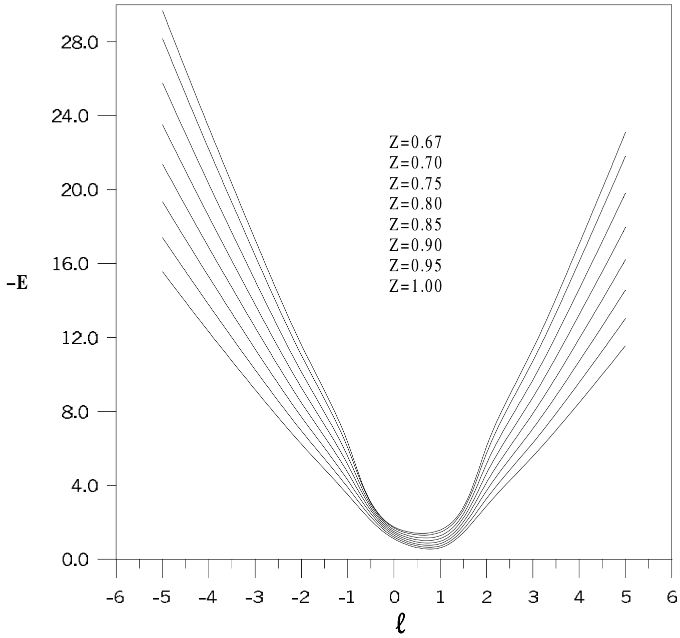

Figure 1. Cubic oscillator (35) eigenenergies $(-E)$ in (44) versus ℓ for $k = 0$ and $|\ell| < 5$ at different values of $Z = Z(\delta) \in (2/3, 1)$.

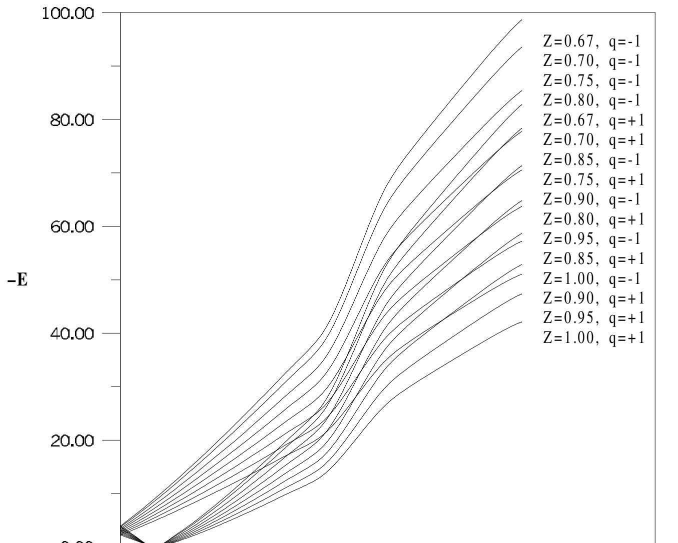

Figure 2. Cubic oscillator (35) eigenenergies $(-E)$ in (44) versus $|\alpha_o|$ for $k = 0, \ell = -1/2 + q|\alpha_o|$, and different values of $Z = Z(\delta) \in (2/3, 1)$ at *even* and *odd quasi-parities*.

that even in the domain of not too large ℓ, the difference between the exact and approximate energies remains small, of the order of $\approx 0.05\%$ from equation (42), with $\beta = 0$, and $\approx 0.2\%$

Table 1. Comparison of the energy levels for model (1) (with $\alpha = 0$). The benchmark, numerically exact DVR values are cited from Znojil *et al* [2].

ℓ	k	DVR	Equation (42)	Equation (44)
5	0	−11.521 91	−11.517	−11.542
	1	−4.564 82	−4.260	−4.900
	2	1.870 17	2.997	−0.109
10	0	−28.765 52	−28.762	−28.776
	1	−20.598 67	−20.426	−20.756
	2	−12.706 40	−12.090	−13.230
	3	−5.116 63	−3.754	−6.380
	4	2.140 32	4.58	−0.727
20	0	−68.726 46	−68.724	−68.733
	1	−59.247 06	−59.149	−59.330
	2	−49.917 73	−49.574	−50.171
	3	−40.745 89	−39.998	−41.283
	4	−31.739 51	−30.423	−32.705
	5	−22.907 12	−20.847	−24.489
	6	−14.257 69	−11.272	−16.713
	7	−5.800 54	−1.696	−9.512
	8	2.454 91	7.879	−3.172
50	0	−211.135 55	−211.134	−211.138
	1	−199.680 09	−199.633	−199.718
	2	−188.294 59	−188.132	−188.406
	3	−176.980 40	−176.631	−177.206
	4	−165.738 89	−165.129	−166.123
	5	−154.571 49	−153.628	−155.161
	6	−143.479 67	−142.127	−144.328
	7	−132.464 94	−130.626	−133.628
	8	−121.528 86	−119.124	−123.069

from equation (44), with $\beta = \beta^{(-)} \neq 0$, for the ground state. Such a prediction should not mislead us in connection with the related convergence/divergence of our energy series (8), which is in fact the genuine test of our present $\mathcal{PT}$-symmetric PSLET formulae. The energy series (8) with $\beta = \beta^{(-)} \neq 0$ converges more rapidly than it does with $\beta = 0$. Nevertheless, our leading energy term remains the benchmark for testing the reality and discreteness of the energy spectrum.

In table 2 we compare our results (using $\beta = \beta^{(-)} \neq 0$, hereinafter, numerically solve for r_0 and following the procedure of section 3) for (1) with $\alpha = 0$ using the first ten terms of (8) and the corresponding Padé approximant, again with those from the DVR approach. They are in almost exact accord. Hereby, we may emphasize that the digital precision is enhanced for larger $\bar{l}$ (smaller $1/\bar{l}$) values, where the energy series (8) and the related Padé approximants stabilize more rapidly.

Extending the recipe of our test beyond the weak-coupling regime $\delta \approx 0$ (i.e. $\alpha \approx 0$) we show, in table 3, the energy dependence on the non-vanishing α. Evidently, the digital precision of our $\mathcal{PT}$-PSLET recipe reappears to be $\bar{l}$-dependent and almost α-independent. In table 4 we witness that the leading energy approximation inherits a substantial amount of the total energy documenting, on the computational and practical methodical side, the usefulness of our pseudo-perturbation recipe beyond its promise of being quite handy. Yet, a broad range of α is considered including the domain of negative values, safely protected against any possible spontaneous $\mathcal{PT}$-symmetry breaking.

Table 2. Same as table 1 with $\mathcal{PT}$-PSLET results from the first ten terms of (8) and the corresponding Padé approximant.

k	ℓ	DVR	$\mathcal{PT}$-PSLET	Padé	$\bar{l} \approx$
0	5	−11.521 91	−11.521 91	−11.521 913 36	4.4
	10	−28.765 52	−28.765 52	−28.765 521 78	9.4
	20	−68.726 46	−68.726 46	−68.726 459 28	19.4
	50	−211.135 55	−211.135 548	−211.135 547 85	49.4
1	5	−4.564 82	−4.565	−4.564 813	2.2
	10	−20.598 67	−20.598 669	−20.598 669 11	7.2
	20	−59.247 06	−59.247 055 56	−59.247 055 572	17.2
	50	−199.680 09	−199.680 086 57	−199.680 086 5747	47.2
2	10	−12.706 40	−12.706 5	−12.796 401 7	4.9
	20	−49.917 73	−49.917 727	−49.917 727 248	14.9
	50	−188.294 59	−188.294 591	−188.294 591 275 07	44.9
3	10	−5.116 63	−9.388	−5.116 6	2.7
	20	−40.745 89	−40.745 89	−40.745 890 26	12.7
	50	−176.980 40	−176.980 399 68	−176.980 399 68	42.7

Table 3. Comparison of the energies for model (1) with $\ell = 10$ and different values of α.

α	k	DVR	$\mathcal{PT}$-PSLET	Padé
20	0	−58.621 90	−58.621 9026	−58.621 902 667
	1	−49.836 26	−49.836 258	−49.836 2579
	2	−41.290 14	−41.290 5	−41.290 14
10	0	−43.852 23	−43.852 233 24	−43.852 233 235
	1	−35.407 17	−35.407 174	−35.407 174 08
	2	−27.230 03	−27.230 10	−27.230 03
0	0	−28.765 52	−28.765 52	−28.765 517 77
	1	−20.598 67	−20.598 67	−20.598 669 108
	2	−12.706 40	−12.706 53	−12.706 4017
−10	0	−13.355 29	−13.355 2878	−13.355 287 796
	1	−5.407 17	−5.407 17	−5.407 173 6
	2	2.276 17	2.08	2.276 17
−20	0	2.381 31	2.381 306	2.381 305 57
	1	10.164 62	10.164 6	10.164 619 1
	2	17.703 39	–	–

6. Summary

Our purpose was to find a suitable *perturbation series like* expansion of the bound states. We have started from the observation that the physical consistency of the model (1) (i.e., the reality of its spectrum) is characterized by the presence of a strongly repulsive/attractive core in the potential $V_p(r)$. This is slightly counterintuitive since the phenomenologically useful values of ℓ are usually small and only the first few lowest angular momenta are relevant in the Hermitian Schrödinger equations with central symmetry.

Only exceptionally, very high partial waves are really needed for phenomenological purposes (say, in nuclear physics [23]). The strong repulsion is required there, first of all, due to its significant *phenomenological* relevance and *in spite* of the formal difficulties.

Table 4. Energies for model (1) with $\ell = 0$ and different values of α, d, k.

d	α	k	Leading term	$\mathcal{PT}$-PSLET	Padé	$\bar{l} \approx$
3	−5	0	3.07	2.903	2.881	1
		1	−1.65	−1.58	−1.579 391	3.2
		2	−7.962	−7.652	−7.655 05	5.4
	−10	0	8.23	7.82	7.820	1.3
		1	3.98	3.803	3.809 92	3.5
		2	−1.87	−1.80	−1.798 903	5.7
	−15	0	13.95	13.38	13.384 19	1.6
		1	9.95	9.52	9.542 833	3.8
		2	4.45	4.27	4.273 163	6.0
1	10	0	−12.46	−12.471	−12.471	1.4
		1	−20.65	−20.304	−20.3040	3.5
		2	−29.00	028.341	−28.340 38	5.7
	5	0	−8.12	−8.082	−8.082	1.5
		1	−15.38	−15.058	−15.057 95	3.6
		2	−23.15	−22.569	−22.568 20	5.9
	−5	0	1.57	1.5393	1.5393	1.8
		1	−4.16	−4.055	−4.055 465	4.0
		2	−10.94	−10.63	−10.634 1055	6.3
	−15	0	12.99	12.754	12.753 91	2.2
		1	8.06	7.84	7.840 06	4.6
		2	1.99	1.93	1.926 9341	6.8

Fortunately, efficient $\ell \gg 1$ approximation techniques already exist for the latter particular realistic Hermitian models. They have been developed by many authors (cf, e.g., their concise review in [24]). Their thorough and critical tests are amply available but similar studies were still missing in the non-Hermitian context.

Our present purpose was to fill the gap at least partially. We have paid thorough attention to the first few open problems related, e.g., to the possible complex deformation of the axis of coordinates. Our thorough study of a few particular $\mathcal{PT}$-symmetric examples revealed that the transition to the non-Hermitian models is unexpectedly smooth. We did not encounter any serious difficulties, in spite of many apparent obstacles as mentioned in section 2 (e.g., an enormous ambiguity of the choice of the most suitable zero-order approximation).

In this way, our present study confirmed that the angular momentum (or dimension) parameter ℓ in the 'strongly spiked' domain where $|\ell| \gg 1$ offers its formal re-interpretation and introduction of an artificial perturbation-like parameter $1/\bar{l}$ which may serve as a guide to the interpretation of many effective potentials as a suitably chosen solvable (harmonic oscillator) zero-order approximation followed by the systematically constructed corrections which prove obtainable quite easily.

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